Summary of Derivative Tests

Note that for all the tests given below it is assumed that the function f is continuous.

Critical Numbers

Definition. A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Critical numbers tell you **where** a **possible** max/min occurs.

First Derivative Test

Use: To find local max/mins

Statement of the test: Suppose c is a critical number.

1. If f' changes from positive to negative at c, then f has a local max at c
2. If f' changes from negative to positive at c, then f has a local min at c
3. If f' is positive to the left and right of c [or negative to the left and right of c], then f has no local max or min at c

Trick to Remember: If you are working on a problem and forget what means what in the above test, draw a picture of tangent lines around a minimum or around a maximum

Steps:

1. Find the critical numbers of f [set $f'(x) = 0$ and solve]
2. Break up the entire number line using the critical points. Then make the table below:

Interval	Test #	$f'(x) =$ ____

3. For each interval choose a test number, plug it into $f'(x)$, and write down the sign of $f'(x)$
4. Use the statement of the test to decide what each critical point is

Remember: The critical points are **where** the max/min occurs. The function value $f(c)$ is **what** the max/min value is.

Finding Intervals of Increase/Decrease

Recall:

- $f'(x) > 0 \implies f$ is increasing
- $f'(x) < 0 \implies f$ is decreasing

Steps:

1. Find the critical numbers of f [set $f'(x) = 0$ and solve]
2. Break up the entire number line using the critical points. Then make the table below:

Interval	Test #	$f'(x) =$ ____	$f(x)$ is ...

3. For each interval choose a test number, plug it into $f'(x)$, and write down the sign of $f'(x)$
4. Use the sign of $f'(x)$ to decide whether $f(x)$ is increasing or decreasing on each interval

Notice that steps (1)-(3) above are exactly the same as the first derivative test. In other words, in order to find the intervals of increase/decrease you need to still do almost all of the same steps as the 1st derivative test. For this reason, if a problem asks you to find the intervals of increase/decrease **and** to find the local mins/maxes, it makes much more since to **use the 1st derivative test**, since you need to do all that work anyways.

Concavity Test

Use: Tells you how to determine when a function is concave up or concave down

Statement of Test:

1. $f''(x) > 0 \implies f$ is concave up
2. $f''(x) < 0 \implies f$ is concave down

Second Derivative Test

Use: To find local max/mins. Easier than the 1st derivative test if you don't need to find intervals of increase/decrease.

Statement of Test: Let c be a critical point of a function $f(x)$. Then

$f'(c)$	$f''(c)$	Critical point is a ...
$= 0$	< 0	max (concave down)
$= 0$	> 0	min (concave up)
$= 0$	$= 0$	inconclusive - use 1st deriv. test

Trick to Remember: If you forget the above test, think about everything in terms of concavity. For example if $f''(c) < 0$ then you know the function is concave down. Drawing what concave down looks like will easily remind you that this is a max.

Steps

1. Find the critical numbers of f [set $f'(x) = 0$ and solve]
2. Find $f''(x)$
3. Plug each critical point into $f''(x)$
4. Use the above table to determine what the critical point is (max or min). If $f''(c) = 0$ use the 1st derivative test.

Finding Intervals of Concavity/Inflection points

The steps for this are basically the exact same steps used for finding intervals of increase/decrease, except you are applying them to the **second derivative**.

1. Find the critical numbers of f' [set $f''(x) = 0$ and solve]
2. Break up the entire number line using the critical points. Then make the table below:

Interval	Test #	$f''(x) =$____	$f(x)$ is concave ...

3. For each interval choose a test number, plug it into $f''(x)$, and write down the sign of $f''(x)$
4. Use the sign of $f''(x)$ to decide whether $f(x)$ is concave up or concave down
5. (Inflection Points) If $f(x)$ c**hanges concavity** at a point c, then you have an **inflection point** at c

Example

Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$. Find intervals of increase/decrease, local max/mins, intervals of concavity, and inflection points.

Intervals of Increase/Decrease
(1) Find Critical Numbers of f [set $f'(x) = 0$]

$$\begin{aligned} f'(x) = 12x^3 - 12x^2 - 24x &= 0 \\ 12x(x^2 - x - 2) &= 0 \\ x = 0,\ x = -1,\ x &= 2 \end{aligned}$$

Critical Numbers: $x = -1, 0, 2$
(2) Break up number line and make table:

Interval	Test #	$f'(x) =$____	$f(x)$ is ...
$x < -1$	$x = -2$	(-)	Decreasing
$-1 < x < 0$	$x = -1/2$	(+)	Increasing
$0 < x < 2$	$x = 1$	(-)	Decreasing
$x > 2$	$x = 3$	(+)	Increasing

Intervals of Increase: $(-1, 0)$ **and** $(2, \infty)$
Intervals of Decrease: $(-\infty, -1)$ **and** $(0, 2)$

Max/Mins
Method 1: First Derivative Test
This method makes the most sense since we already have the table above.
$x = -1 : f'(x)$ goes from $(-)$ to $(+)$ $\implies$ local min at $x = -1$ and the value is $f(-1) = 0$
$x = 0 : f'(x)$ goes from $(+)$ to $(-)$ $\implies$ local max at $x = 0$ and the value is $f(0) = 5$
$x = 2 : f'(x)$ goes from $(-)$ to $(+)$ $\implies$ local min at $x = 2$ and the value is $f(2) = -27$

Method 2: Second Derivative Test
This is just to show you how to use the second derivative test and that you get the same answer.

We already found the critical numbers $x = -1, 0, 2$. So we just need the second derivative:

$$f''(x) = 36x^2 - 24x - 24$$

Now plug each critical point into the second derivative and make a conclusion:

$x = -1 : f''(-1) = 36 + 24 - 24 > 0 \implies$ local min at $x = -1$ (concave up)
$x = 0 : f''(0) = 0 - 0 - 24 < 0 \implies$ local max at $x = 0$ (concave down)
$x = 2 : f''(2) > 0 \implies$ local min at $x = 2$ (concave up)

Intervals of Concavity

(1) Find the critical numbers of f' [set $f''(x) = 0$]

$$f''(x) = 36x^2 - 24x - 24 = 0$$
$$3x^2 - 2x - 2 = 0$$
$$x = \frac{2 \pm \sqrt{4 - 4(3)(-2)}}{2(3)}$$
$$= \frac{2 \pm \sqrt{28}}{6}$$

So $x \approx -0.55$ and $x \approx 1.22$

(2) Break up the entire number line using the critical points.

Interval	Test #	$f''(x) =$____	$f(x)$ is concave ...
$x < -0.55$	-1	$f''(-1) = 36 + 24 - 24 > 0$	up
$-0.55 < x < 1.22$	0	$f''(0) = -24 < 0$	down
$x > 1.22$	2	$f''(2) > 0$	up

Concave Up: $\left(-\infty, \frac{2-\sqrt{28}}{6}\right)$ **and** $\left(\frac{2+\sqrt{28}}{6}, \infty\right)$

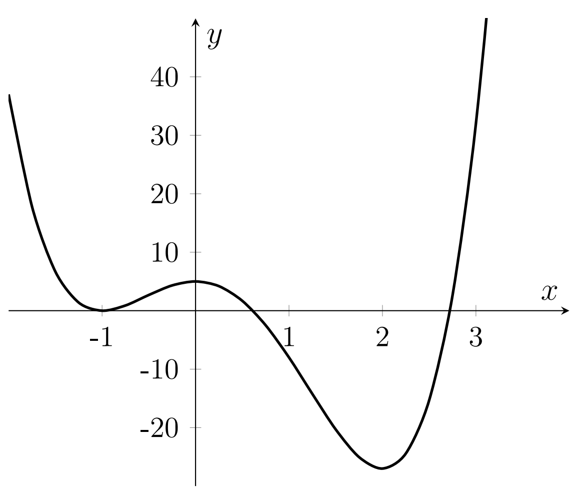
Concave Down: $(\frac{2-\sqrt{28}}{6}, \frac{2+\sqrt{28}}{6})$

Inflection Points

Using the same table as we did for concavity, we see that $f(x)$ changes concavity twice: first at $x = \frac{2-\sqrt{28}}{6}$ from up to down and second at $x = \frac{2+\sqrt{28}}{6}$ from down to up. Hence, both of these are inflection points.

Inflection Points: $x = \frac{2-\sqrt{28}}{6}$ **and** $x = \frac{2+\sqrt{28}}{6}$

Graph of $f(x)$: Using all the information above you can draw a complete graph of $f(x)$



Closed Interval Method

Use: To find **absolute** mins/maxes on an interval $[a, b]$

Steps:

1. Find the critical numbers of f [set $f'(x) = 0$ and solve]
2. Find the value of f at each critical number
3. Find the value of f at each endpoint [i.e., find $f(a)$ and $f(b)$]
4. Compare all of the above **function values**. The largest one is the absolute max and the smallest one is the absolute min.