Section 5.4: Curve Sketching

Checklist for Graphing a Function

A. Use $f(x)$ to

1. Determine the domain of the function and the intervals on which the function is continuous.
2. ~~Determine whether the function is symmetric about the y-axis or the origin.~~
3. Find all vertical asymptotes.
4. Find all horizontal asymptotes.
5. Find where the function crosses the axes.

B. Use $f'(x)$ to

6. Find the critical values.
7. Find intervals where the function is increasing and decreasing.
8. Find all relative extrema.

C. Use $f''(x)$ to

9. Find intervals where the graph of the function is concave up and concave down.
10. Find all inflection values.

D. (Final step.) Use Steps A, B, C, and the values of f at the critical values and inflection values to graph.

Ex: $f(x) = \sqrt{x^2+1} - x$

A. (1) Domain $= \mathbb{R}$, continuous $= \mathbb{R}$

(2) Find VA's:

NO vertical since f is not infinite at any x-value (no denom. which $\to 0$)

(3) Find HA's:

To find HA, take

$$\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) = \infty - \infty \quad * \text{ need more algebra}$$

$$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+1} - x\right)}{1}\left(\frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x}\right)$$

$$= \lim_{x\to\infty} \frac{(x^2+1) - x^2}{\sqrt{x^2+1} + x}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1} + x} = 0$$

Same for $\lim_{x\to -\infty} f(x)$.

$\Rightarrow$ HA at $y = 0$ as $x \to \pm\infty$.

(4) Find out where curve crosses axes:

y-axis $(x=0)$: $f(0) = 1 - 0 = 1$

$\Rightarrow$ crosses y-axis at $(0,1)$

; x-axis $(y=0)$: $\sqrt{x^2+1} - x = 0$

$\Rightarrow x^2+1 = x$

$\Rightarrow 1 = 0$

$\Rightarrow$ never crosses x-axis

B. (5) Find Critical #'s:

$$f'(x) = \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x - 1 = \frac{x}{\sqrt{x^2+1}} - 1 = 0 \Rightarrow x - \sqrt{x^2+1} = 0 \Rightarrow \text{impossible}$$

* no critical points

(6) Find intervals of increase / decrease

* Since there are no CP's, just consider the whole real line

interval	test #	$f'(x)$	$f(x)$ is...
$(-\infty, \infty)$	$x=0$	$0-1=-1$	Decreasing

(makes sense since $f'(x) = x - \sqrt{x^2+1} < 0$ for all x)

Decreasing: $(-\infty, \infty)$

C. (7) Find intervals of concavity + inflection points

$$f'(x) = \frac{x}{\sqrt{x^2+1}} - 1 \Rightarrow f''(x) = \frac{\sqrt{x^2+1} - x \cdot \frac{1}{2}(x^2+1)^{-1/2}(2x)}{(\sqrt{x^2+1})^2}$$

$$= \frac{\sqrt{x^2+1} - x^2/\sqrt{x^2+1}}{x^2+1}$$

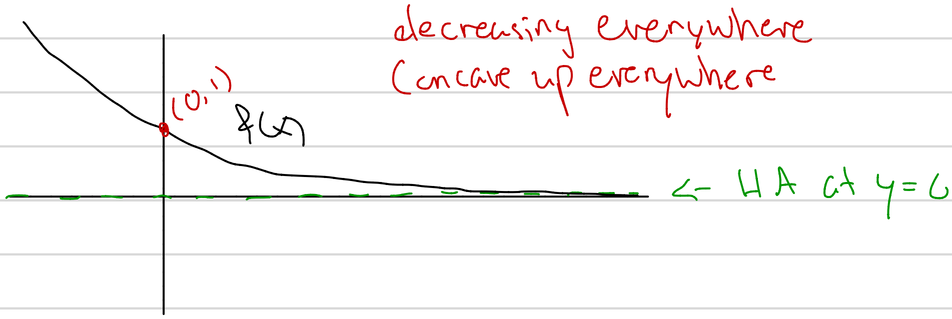
$$= \frac{\frac{(x^2+1) - x^2}{\sqrt{x^2+1}}}{x^2+1}$$

$$= \frac{1}{(x^2+1)^{3/2}}$$

$f''(x) = \frac{1}{(x^2+1)^{3/2}}$ is always positive + never zero. (similar to above)

$\Rightarrow f(x)$ is concave up on $(-\infty, \infty)$ + no inflection points

D. Use (a) - (c) to sketch graph



Ex $f(x) = \frac{x^2}{x^2-1}$

A. (1) Domain $= \mathbb{R}, x \neq \pm 1$, Cont. $= \mathbb{R}, x \neq \pm 1$

(2) Find VAs;

$\lim_{x \to -1^-} f(x) = +\infty$, $\lim_{x \to -1^+} f(x) = -\infty$ | $\lim_{x \to 1^+} f(x) = -\infty$, $\lim_{x \to 1^-} f(x) = +\infty$

$\Rightarrow$ VA at $x=1$, $x=-1$

(3) Find HAs:

$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{1 - 1/x^2} = 1$, $\lim_{x \to -\infty} f(x) = 1$

$\Rightarrow$ HA at $y = 1$

(4) Cross axes:

y-axis $(x=0)$: $f(0) = \frac{0}{0-1} = 0$. $\Rightarrow$ crosses y-axis at $(0,0)$

x-axis $(y=0)$: $\frac{x^2}{x^2-1} = 0 \Rightarrow x^2=0 \Rightarrow x=0 \Rightarrow$ crosses x-axis at $(0,0)$

B. (5) Find Critical #'s

$$f'(x) = \frac{2x(x^2-1) - x^2 \cdot 2x}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2} = 0 \Rightarrow x=0$$

(6) Find intervals of increase/decrease

interval	test #	$f'(x)$	$f(x)$ is...
$(-\infty,-1)$	-2	$+$	increasing
$(-1,0)$	$-\frac{1}{2}$	$+$	increasing
$(0,1)$	$\frac{1}{2}$	$-$	decreasing
$(1,\infty)$	2	$-$	decreasing

increasing: $(-\infty,-1) \cup (-1,0)$
decreasing: $(0,1) \cup (1,\infty)$

c. (7) Find intervals of concavity + inflection points

$$f''(x) = \frac{-2(x^2-1)^2 - (-2x)\cdot 2(x^2-1)\cdot 2x}{(x^2-1)^4}$$

$$= \frac{6x^2+2}{(x^2-1)^3} = 0$$

$\Rightarrow 6x^2+2=0$

$\Rightarrow 3x^2=-2$

$\Rightarrow$ no solution

$\Rightarrow$ only need to consider when denom. = 0

interval	test #	$f''(x)$	$f(x)$ is
$(-\infty,-1)$	-2	$+$	concave up
$(-1,1)$	0	$-$	concave down
$(1,\infty)$	2	$+$	concave up

concave up: $(-\infty,-1) \cup (1,\infty)$

concave down: $(-1,1)$

No inflection pnts since $x=\pm 1$ are undefined