Part I Definitions and Concepts

1. Given a function $f(x)$, what is the equation for the linearization $L(x)$ of the function at a? [2]

$L(x) = f'(a)(x-a) + f(a)$

2. Population growth can be modeled by the differential equation $\frac{dP}{dt} = kP(t)$, where $P(t)$ is the population after t years and k is a constant. The solution to this differential equation is [2]

$$P(t) = P(0)e^{kt}.$$

In this equation,

(a) What does $P(0)$ represent?

1 Initial condition - Population at time $t=0$

(b) What does k represent?

1 Average growth rate

Part II Calculations

1. Find the derivative of the following functions.

(a) $y = 2^{x^3}$ [5]

2 1 1

$y' = 2^{x^3} \ln(2) \cdot \frac{d}{dx}(x^3)$

$= 3x^2\, 2^{x^3} \ln(2)$

1

OR

1 1 constant

$\ln(y) = \ln(2^{x^3}) = x^3 \ln(2)$

1 $\frac{y'}{y} = 3x^2 \ln(2)$ 1

$y' = y\, 3x^2 \ln(2) = 2^{x^3}\, 3x^2 \ln(2)$ 1

(b) $f(x) = \dfrac{\ln^2(x)}{x} = \dfrac{(\ln(x))^2}{x}$ [5]

$f = (\ln(x))^2 \qquad g = x$ 1

2 $f' = 2\ln(x) \cdot \frac{1}{x} \qquad g' = 1$ 1

$$f'(x) = \frac{(2\ln(x)\frac{1}{x})(x) - (\ln(x))^2(1)}{x^2}$$ 1

$$= \frac{2\ln(x) - \ln^2(x)}{x^2}$$

2. Use implicit differentiation to find the the derivative of $y^2 - x^2 = 2xy$. [5]

Product Rule

$f = 2x \qquad g = y$

$f' = 2 \qquad g' = y'$

$$\frac{d}{dx}(y^2 - x^2) = \frac{d}{dx}(2xy)$$

1 1 1 1

$$\Rightarrow 2yy' - 2x = 2y + 2xy'$$

$$\Rightarrow 2yy' - 2xy' = 2y + 2x$$

$$\Rightarrow y'(2y - 2x) = 2y + 2x$$

$$\Rightarrow y' = \frac{2y + 2x}{2y - 2x} = \frac{y + x}{y - x}$$ 1

3. The half-life of cesium-137 is 30 years. Given an initial sample of 70-mg, find a formula for the mass of the sample that remains after t years. [5]

General Formula: $m(t) = m(0)e^{kt}$ 1

$m(0) = 70$

$m(30) = \frac{1}{2}m(0)$ 1

$= 35$

$m(t) = 70e^{kt}$

$m(30) = 70e^{30k} = 35$ 1

$e^{30k} = 35/70$

$30k = \ln(35/70)$

1 $k = \frac{1}{30}\ln(35/70)$

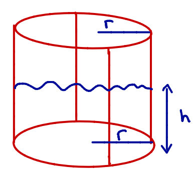
$$\boxed{m(t) = 70e^{\frac{1}{30}\ln(35/70)t}}$$

1

$= 70e^{1/30 \ln(1/2)t}$

$= 70e^{-\ln(2)/30}$

4. A cylindrical tank with radius 3 m is being filled with water at a rate of 4 m^3/min. How fast is the height of the water increasing? Recall that the volume of a cylinder is $V = \pi r^2 h$. [5]



Given: $\frac{dV}{dt} = 4\ m^3/min$

$r = 3m$ 1

The radius of a cylinder is constant.

$V = \pi(3)^2 h = 9\pi h$ 1

$\Rightarrow \frac{dV}{dt} = 9\pi\frac{dh}{dt}$ 2

$\Rightarrow \frac{dh}{dt} = \frac{4}{9\pi}$ m/min. 1

5. Let $g(x) = \sqrt[3]{1+x}$. [6]

(a) Find the linear approximation of $g(x)$ at $a = 0$.

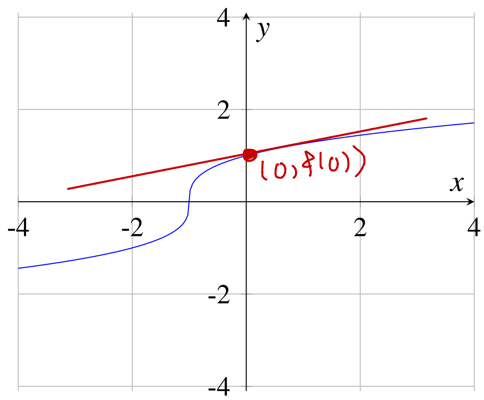
$L(x) = g'(a)(x-a) + g(a) = g'(0)(x-0) + g(0)$ 1

$g'(x) = \frac{1}{3}(1+x)^{-2/3} \Rightarrow g'(0) = \frac{1}{3}$ 1

$g(0) = 1$

$L(x) = \frac{1}{3}(x-0) + 1 = \frac{1}{3}x + 1$ 1

(b) Illustrate your linear approximation by graphing the tangent line on the graph of $g(x)$ below.



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(c) Use the linear approximation you found in part (a) to approximate the number $\sqrt[3]{1.1}$.

$g(x) = \sqrt[3]{1+x} \Rightarrow$ we need $x = 0.1$ 1

$\Rightarrow g(0.1) = \sqrt[3]{1.1}$

$L(.1) = \frac{1}{3}(.1) + 1 = \frac{31}{30} \approx 1.0333$ 1