Section 4.7: Optimization Solutions

- Methods for finding extreme values have many practical applications
- Often the hardest part is turning English into math

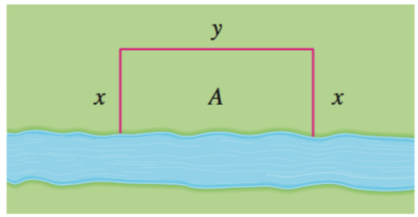
Problem Solving Tips (similar to 3.7):

- Step 1: Gather information about the problem
  - Draw a diagram
  - Identify formulas
  - Figure out what the constraint is using the given information in the problem
  - Identify what formula is being maximized and minimized
- Step 2: Write down formulas
  - Carefully translate between english and mathematics
  - One formula will involve the constraint
  - The other formula will be what you want to maximize/minimize
- Step 3: Substitute Into the Formula You Want to Maximize/Minimize
  - Use the constraint to solve for one of the variables
  - Substitute into the formula you want to maximize/minimize so that now you only have **one variable**
- Step 4: Find critical numbers
- Step 5: Test the critical numbers to identify if they are a min/max
  - Usually the easiest way to do this is by using the **second derivative test**
- Step 6: Answer the Question
  - Make sure to go back and answer what is being asked. It is usually *not* the case that the answer you got in Step 5 is actually the answer to the original question.

Example 1. A farmer has 2400 ft. of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fencing along the river. What are the dimensions of the field that has the largest area?

Step 1: **Gather Information**

(a) Make a diagram:



(b) What formulas will be used in this problem? **Area of a rectangle** and **Length of Fencing**

(c) Figure out what the constraint is:

$$L = 2400 \text{ ft.}$$

(d) What do you want to maximize/minimize? **Area**

Step 2: Write Down Formulas

$$A = x \cdot y \text{ and } L = 2x + y = 2400$$

Step 3: Substitute Into the Formula You Want to Maximize/Minimize

$$2x + y = 2400 \implies y = 2400 - 2x$$

$$A = x \cdot y = x(2400 - 2x) = 2400x - 2x^2$$

Step 4: Find Critical Numbers

$$A(x) = 2400x - 2x^2$$

$$A'(x) = 2400 - 4x = 0 \implies x = 600.$$

Step 5: Test Critical Numbers

$$A''(x) = -4 < 0 \text{ for all } x \implies x = 600 \text{ is a max}$$

Step 6: Answer The Question

The question was "What are the dimensions of the field that has the largest area?". The largest area occurs when $x = 600$ ft. We need to find y:

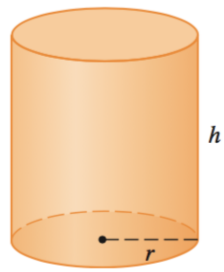
$$y = 2400 - 2x = 2400 - 2(600) = 1200 \text{ ft.}$$

Hence, the dimensions of the field is 600 ft $\times$ 1200 ft.

Example 2. A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Step 1: **Gather Information**

(a) Make a diagram:



where r = radius (cm) and h = height (cm).

(b) What formulas will be used in this problem? **Surface Area of Cylinder** and **Volume of Cylinder**

(c) Figure out what the constraint is:

$$V = 1 \text{ L} = 1000 \text{ cm}^3$$

(d) What do you want to maximize/minimize? **Surface Area**

Step 2: Write Down Formulas

$$\begin{aligned} A &= \textbf{Area of two circles} + \textbf{ Area of cylinder} \\ &= 2\pi r^2 + 2\pi r h \end{aligned}$$

$$V = \pi r^2 h = 1000$$

Step 3: Substitute Into the Formula You Want to Maximize/Minimize

$$\pi r^2 h = 1000 \implies h = \frac{1000}{\pi r^2}$$

$$A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

Step 4: Find Critical Numbers

$$A(r) = 2\pi r^2 + \frac{2000}{r}$$

$$\begin{aligned} A'(r) = 4\pi r - \frac{2000}{r^2} &= \frac{4\pi r^3 - 2000}{r^2} = 0 \\ \implies 4\pi r^3 - 2000 &= 0 \\ \pi r^3 &= 500 \\ r &= \sqrt[3]{500/\pi} \end{aligned}$$

Step 5: Test Critical Numbers

$$A''(r) = 4\pi + \frac{4000}{r^3} \implies A''\left(\sqrt[3]{\frac{500}{\pi}}\right) > 0 \implies r = \sqrt[3]{500/\pi} \text{ is a min.}$$

Step 6: Answer The Question

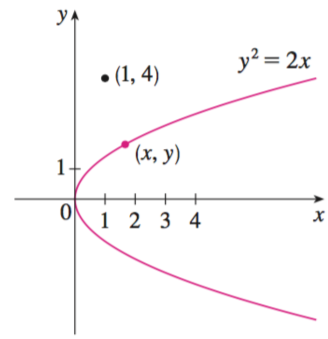
The question was "Find the dimensions that will minimize the cost of the metal to manufacture the can." We have that $r = \sqrt[3]{500/\pi}$ minimizes the surface area. We need to find h:

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi(500/\pi)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r.$$

Example 3. Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$. *Recall*: The distance between two points is given by the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Step 1: Gather Information

(a) Make a diagram:



(b) What formulas will be used in this problem? **The Equation of the Parabola** and **Distance Formula**

(c) Figure out what the constraint is:

$$y^2 = 2x$$

(d) What do you want to maximize/minimize? **Distance**

Step 2: Write Down Formulas

$$y^2 = 2x$$

$$d = \sqrt{(x-1) + (y-4)^2} = \textbf{Distance between } (1, 4) \textbf{ and } (x, y)$$

Step 3: Substitute Into the Formula You Want to Maximize/Minimize

$$y^2 = 2x \implies x = \frac{y^2}{2}$$

$$d = \sqrt{(x-1)^2 + (y-4)^2} = \sqrt{(y^2/2 - 1)^2 + (y-4)^2}$$

It turns out that the minimum of d occurs at the same point as the minimum of d^2 (convince yourself of this), but d^2 is easier to work with:

$$f(y) = d^2 = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2$$

Step 4: Find Critical Numbers

$$f(y) = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2$$

$$\begin{aligned} f'(y) = 2\left(\frac{y^2}{2} - 1\right) y + 2(y-4) = y^3 - 8 = 0 \\ \implies y^3 = 8 \\ y = 2 \end{aligned}$$

Step 5: Test Critical Numbers

$$f''(y) = 3y^2 > 0 \text{ for all } y \implies 2 \text{ is a min.}$$

Step 6: Answer The Question

The question was "Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1,4)$." We found that $y = 2$ minimizes the distance. We need to find x:

$$x = \frac{y^2}{2} = \frac{(2)^2}{2} = 2.$$

Hence, the point (2,2) minimizes the distance between the parabola $y^2 = 2x$ and the point (1,4).

Example 4. A rectangular box with an open top is to have a volume of 60 m^3. The length of the base is twice the width. Material for the base costs \$10/m^2. Material for the sides cost \$6/m^2. Find the cost of materials for the cheapest container.

Step 1: Gather Information

(a) Make a diagram:

(b) What formulas will be used in this problem? **Volume of a Rectangle** and **Cost of Materials**

(c) Figure out what the constraint is:

$$V = 60$$

(d) What do you want to maximize/minimize? **Material Cost**

Step 2: Write Down Formulas

$$l = 2w$$

$$V = lwh = (2w)(w)h = 2w^2h = 60$$

$$\begin{aligned} C &= \textbf{Cost of Base} \times \textbf{Area of Base} + \textbf{Cost of sides} \times \textbf{Area of Sides} \\ &= 10 \times l \cdot w + 6 \times l \cdot h \times 2 + 6 \times w \times h \times 2 \\ &= 10(2w)w + 12(2w)h + 12wh \\ &= 20w^2 + 36wh \end{aligned}$$

Step 3: Substitute Into the Formula You Want to Maximize/Minimize

$$V = 2w^2h = 60 \implies h = 60/2w^2 = 30/w^2$$

$$\begin{aligned} C &= 20w^2 + 36wh \\ &= 20w^2 + 36w\left(\frac{30}{w^2}\right) \\ &= 20w^2 + \frac{1080}{w} \end{aligned}$$

Step 4: Find Critical Numbers

$$C(w) = 20w^2 + \frac{1080}{w}$$

$$\begin{aligned}
C'(w) &= 40w - \frac{1080}{w^2} = 40\frac{(w^3-27)}{w^2} = 0 \\
&\implies \left(w^3 - 27\right) = 0 \\
&\implies w = \sqrt[3]{27} = 3.
\end{aligned}$$

Step 5: Test Critical Numbers

Taking the second derivative is more difficult than the past problems. Use first derivative test instead. Choose test points to the left and right of $w = 3$:

$$\begin{aligned}
w = 1: \quad C'(1) &= \frac{40(-26)}{1^2} < 0 \\
w = 4: \quad C'(4) &= \frac{40(40^3 - 27)}{40^2} > 0
\end{aligned}$$

Hence, $w = 3$ is a minimum. [Alternatively, you *could* take the second derivative as before, but notice it involves the quotient rule.]

Step 6: Answer The Question

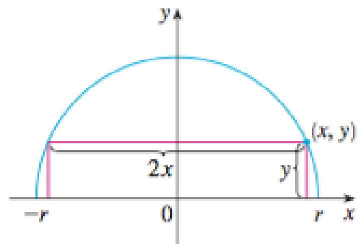
The question was "Find the cost of materials for the cheapest container." So just substitute $w = 3$ into $C(w)$:

$$C(3) = 20(3)^2 + \frac{1080}{3} = \$540.$$

Example 5. Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

Step 1: Gather Information

(a) Make a diagram:



(b) What formulas will be used in this problem? **Area of a Rectangle** and **Equation of a Circle**

(c) Figure out what the constraint is:

$$x^2 + y^2 = r^2$$

(d) What do you want to maximize/minimize? **Area of Rectangle**

Step 2: Write Down Formulas

$$A = 2xy$$

$$x^2 + y^2 = r^2$$

Step 3: Substitute Into the Formula You Want to Maximize/Minimize

$$x^2 + y^2 = r^2 \implies y = \sqrt{r^2 - x^2}$$
$$A = 2xy = 2x\sqrt{r^2 - x^2}$$

Step 4: Find Critical Numbers

$$A(x) = 2x\sqrt{r^2 - x^2}$$

$$A'(x) = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}} = 0$$
$$\implies 2(r^2 - 2x^2) = 0$$
$$\implies 2x^2 = 4r^2$$
$$\implies x = \pm\frac{r}{\sqrt{2}}$$
$$\implies x = \frac{r}{\sqrt{2}} \text{ [since } x \geq 0]$$

Step 5: Test Critical Numbers

$$A'(x) = 2r^2 - 4x^2 \implies A''(x) = -8x$$

$$A''\left(\frac{r}{\sqrt{2}}\right) = -8\left(\frac{r}{\sqrt{2}}\right) < 0 \text{ since } r > 0$$

Hence, $x = \dfrac{r}{\sqrt{2}}$ is a maximum.

Step 6: Answer The Question

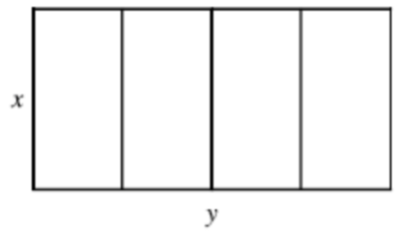
The question was "Find the area of the largest rectangle" So just substitute $x = \dfrac{r}{\sqrt{2}}$ into $A(x)$:

$$\begin{aligned} A\left(\frac{r}{\sqrt{2}}\right) &= 2\left(\frac{r}{\sqrt{2}}\right)\sqrt{r^2 - \left(\frac{r}{\sqrt{2}}\right)^2} \\ &= \frac{2r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}} \\ &= \frac{2r}{\sqrt{2}}\sqrt{\frac{r^2}{2}} \\ &= \frac{2r}{\sqrt{2}}\left(\frac{r}{\sqrt{2}}\right) \\ &= r^2. \end{aligned}$$

Example 6. A farmer with 950 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?

Step 1: **Gather Information**

(a) Make a diagram:



(b) What formulas will be used in this problem? **Area of a Rectangle** and **Length of Fencing**

(c) Figure out what the constraint is:

$$L = 950$$

(d) What do you want to maximize/minimize? **Area of Rectangle**

Step 2: Write Down Formulas

$$A = xy$$

$$L = 5x + 2y = 950$$

Step 3: Substitute Into the Formula You Want to Maximize/Minimize

$$5x + 2y = 950 \implies y = 475 - \frac{5}{2}x$$

$$A = xy = x\left(475 - \frac{5}{2}x\right) = 475x - \frac{5}{2}x^2$$

Step 4: Find Critical Numbers

$$A(x) = 475x - \frac{5}{2}x^2$$

$$A'(x) = 475 - 5x = 0$$

$$\implies 5x = 475$$

$$\implies x = 95$$

Step 5: Test Critical Numbers

Second Derivative Test:

$$A''(x) = -5 < 0 \text{ for all } x$$

Hence, $x = 95$ is a maximum.

Step 6: Answer The Question

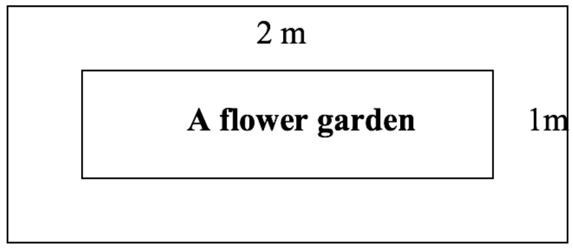
The question was "What is the largest possible total area of the four pens?" So just substitute $x = 95$ into $A(x)$:

$$\begin{aligned} A(95) &= 475(95) - \frac{5}{2}(95)^2 \\ &= 22,562.5 \text{ ft}^2 \end{aligned}$$

Example 7. A rectangular flower garden with an area of 30 m^2 is surrounded by a grass border 1 m wide on two sides and 2 m wide on the other two sides. What dimensions of the garden minimize the combined area of the garden and borders?

Step 1: **Gather Information**

(a) Make a diagram:



(b) What formulas will be used in this problem? **Area of Garden** and **Combined Area of Garden and Borders**

(c) Figure out what the constraint is:

$$G = 30$$

(d) What do you want to maximize/minimize? **Combined Area**

Step 2: Write Down Formulas

$$G = xy = 30$$
$$C = (x+4)(y+2)$$

Step 3: Substitute Into the Formula You Want to Maximize/Minimize

$$xy = 30 \implies y = \frac{30}{x}$$

$$\begin{aligned} C &= (x+4)(y+2) \\ &= (x+4)\left(\frac{30}{x}+2\right) \\ &= 30 + 2x + \frac{120}{x} + 8 \\ &= 2x + \frac{120}{x} + 38 \end{aligned}$$

Step 4: Find Critical Numbers

$$C(x) = 2x + \frac{120}{x} + 38$$

$$\begin{aligned} C'(x) = 2 - \frac{120}{x^2} = \frac{2x^2 - 120}{x^2} = 0 \\ \implies 2x^2 - 120 = 0 \ \text{[since } x > 0\text{]} \\ \implies x^2 = 60 \\ \implies x = \sqrt{60} \end{aligned}$$

Step 5: Test Critical Numbers

Second Derivative Test:

$$\begin{aligned} C''(x) = \frac{240}{x^3} \\ \implies C''(\sqrt{60}) > 0 \end{aligned}$$

Hence, $x = \sqrt{60}$ is a maximum.

Step 6: Answer The Question

The question was "What dimensions of the garden..." We have $x = \sqrt{60}$ m. So substitute into y:

$$y = \frac{30}{x} = \frac{30}{\sqrt{60}} \text{ m}$$