Exam 2

Section W10 - Summer Session I 2016

June 22, 2016

Name: SOLUTIONS

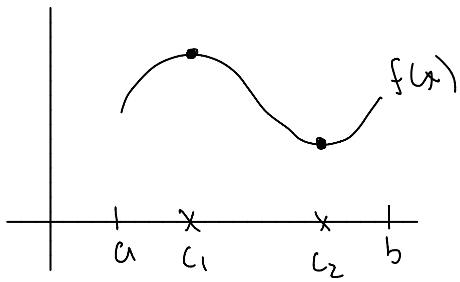
PLEASE READ THE FOLLOWING INFORMATION.

- Please read each question carefully. Show **all** work clearly in the space provided. In order to receive full credit on a problem, solution methods must be complete, logical and understandable.
- Answers must be clearly labeled in the spaces provided after each question. Please cross out or fully erase any work that you do not want graded. The point value of each question is indicated after its statement.
- Give any numerical answers in exact form, not as approximations. For example, one-third is $\frac{1}{3}$, not .33 or .33333. And one-half of π is $\frac{1}{2}\pi$, not 1.57 or 1.57079.
- No books or other references are permitted. Turn smart phones, cell phones, and other electronic devices off and store them away.
- Calculators are allowed but you must show all your work in order to receive credit on the problem.
- If you finish early then you can hand in your exam early.

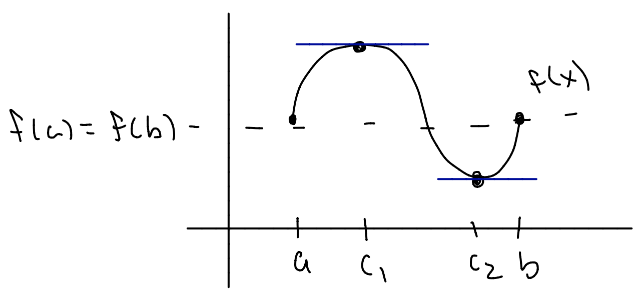
Part I Definitions and Concepts

1. Below are three theorems about continuous function. Draw a picture that illustrates each theorem, using the notation of the theorem in your picture.

   (a) **Extreme Value Theorem:** If $f(x)$ is a continuous function on $[a,b]$ then it has an absolute maximum value and an absolute minimum value on $[a,b]$. [2]

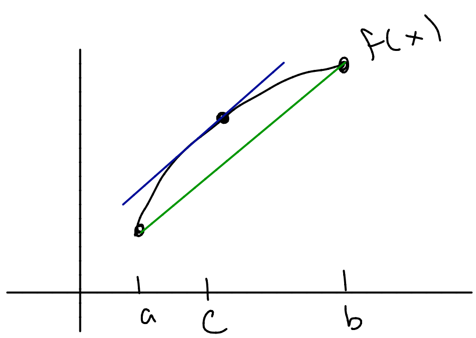


   (b) **Rolle's Theorem:** If $f(x)$ is a continuous function on $[a,b]$ that is differentiable on (a,b), and $f(a) = f(b)$, there is a $c \in (a,b)$ such that $f'(c) = 0$. [2]



   (c) **Mean Value Theorem:** If $f(x)$ is a continuous function on $[a,b]$ that is differentiable on (a,b), there is a $c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$. [2]

   Draw a picture for the case $f(a) \neq f(b)$.



2. What is the definition of a **critical point**? [2]

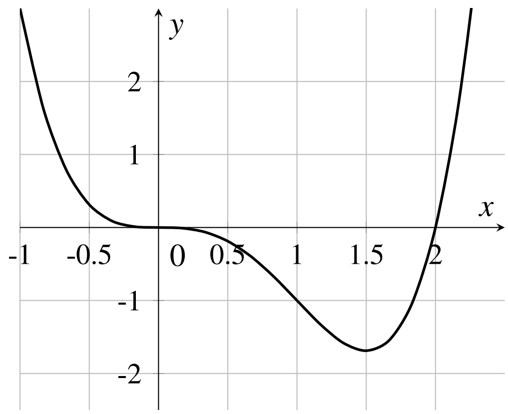
A critical point is a point $(c, f(c))$, where c is a number in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

1 1

3. What is the definition of an **absolute maximum** and an **absolute minimum**? [4]

- Absolute Maximum -
The absolute largest height that occurs for **all** $x-$values under consideration.
1 1
- Absolute Minimum -
The absolute smallest height that occurs for **all** $x-$values under consideration.
1 1

4. Use the graph below to answer the following questions. [4]



(a) The coordinates (x, y) of all the local minimum points are (if there are none, write "none"):
$\approx (1.5, -1.5)$ 1

(b) The coordinates (x, y) of a;; the local maximum points are (if there are none, write "none"):
None 1

(c) The coordinates (x, y) of all the points of inflections are (if there are none, write "none"):
$(0,0)$ and $(1, -1)$
1 1

5. Fill in the following tables. [5]

Concavity Test	
If $f''(x)$ is...	then $f(x)$ is...
Positive	concave up
Negative	concave down

Second Derivative Test		
If $f'(c)$	and $f''(c)$	then the point c is...
$=0$	<0	a local max
$=0$	>0	a local min
$=0$	$=0$	inconclusive

Part II Calculations

1. Find the derivative of the following functions. You do not need to simplify.

(a) $f(x) = \dfrac{\ln^2(x)}{e^x - 1}$ [4]

2 1

$$f'(x) = \frac{2\ln(x)x^{-1} \cdot (e^x - 1) - \ln^2(x)e^x}{(e^x - 1)^2}$$

1

(b) $f(x) = 5^{x^3}$ [4]

2 2

$$f'(x) = 5^{x^3}\ln(5) \cdot 3x^2$$

(c) $f(x) = x^{\sqrt{x}}$ [4]

Let $y = x^{\sqrt{x}}$. Taking the ln of both sides gives you

$$\ln(y) = \ln(x^{\sqrt{x}}) = \sqrt{x}\ln(x) \quad 1$$

Taking the derivative of both sides yields

$$1 \quad \frac{y'}{y} = \frac{1}{2}x^{-1/2}\ln(x) + \sqrt{x}\frac{1}{x} \quad 1$$

Solving for y' gives you

$$y' = x^{\sqrt{x}}\left(\frac{1}{2}x^{-1/2}\ln(x) + \sqrt{x}\frac{1}{x}\right) \quad 1$$

(d) $f(x) = \ln(1+\cos(2x))$ [4]

$$f'(x) = \overset{1}{\frac{1}{1+\cos(2x)}} \cdot (\overset{2}{-\sin(2x)} \cdot \overset{1}{2})$$

(e) $f(x) = \tan^2(\sqrt{x})$ [4]

1 2 1

$$f'(x) = 2\tan(\sqrt{x}) \cdot \sec^2(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2}$$

2. Use implicit differentiation to find the equation of the tangent line to the graph $y^2 = x^3 + 2xy$ at the point $(3, -3)$. Write the final answer in the form $y = mx + b$. [7]

1 1 2

$$\begin{aligned}
2yy' &= 3x^2 + 2[1 \cdot y + x \cdot y'] \\
2yy' &= 3x^2 + 2y + 2xy' \\
2yy' - 2xy' &= 3x^2 + 2y \\
y'(2y - 2x) &= 3x^2 + 2y \\
y' &= \frac{3x^2 + 2y}{(2y - 2x)} \quad 1
\end{aligned}$$

So at the point $(3, -3)$ we have that

$$y' = \frac{27 - 6}{-6 - 6} = -\frac{21}{12} = -\frac{7}{4} \quad 1$$

Hence,

$$\begin{aligned}
y - (-3) &= -\frac{7}{4}(x - 3) \\
y &= -\frac{7}{4}x + \frac{9}{4} \quad 1
\end{aligned}$$

3. Let $f(x) = \sqrt{1-x}$.

(a) Find the linearization of $f(x)$ at $a = 0$. [5]

$$L(x) = f'(a)(x-a) + f(a) = f'(0)(x-0) + f(0) \quad 1$$

$$1 \quad f'(x) = \frac{1}{2}(1-x) \cdot -1 \implies f'(0) = -\frac{1}{2} \quad 1$$

$$f(0) = \sqrt{1-0} = 1 \quad 1$$

$$L(x) = -\frac{1}{2}x + 1 \quad 1$$

(b) Use the linear approximation obtained in part (a) to approximate $\sqrt{0.9}$. [3]

$$\overset{1}{x = 0.1} \implies f(0.1) = \sqrt{1-0.1} = \sqrt{0.9}$$

$$\underset{1}{L(0.1)} = -\frac{1}{2}(0.1) + 1 = \underset{1}{0.95}$$

4. Find the linearization of the function $f(x) = \sqrt[3]{1+x^2}$ at $a = 1$. [5]

$$L(x) = f'(a)(x-a) + f(a) = f'(1)(x-1) + f(1) \quad 1$$

$$1 \quad f'(x) = \frac{1}{3}(1+x^2)^{-2/3}(2x) \implies f'(1) = \frac{1}{3}(2)^{-2/3}(2) = \frac{\sqrt[3]{2}}{3} \quad 1$$

$$f(1) = \sqrt[3]{1+1} = \sqrt[3]{2} \quad 1$$

$$L(x) = \frac{\sqrt[3]{2}}{3}(x-1) + \sqrt[3]{2} = \frac{\sqrt[3]{2}}{3}x + \frac{2\sqrt[3]{2}}{3} \quad 1$$

5. The element Unobtanium has a half-life of 5 years.

(a) Suppose that you have an initial 800 kg mound of Unobtanium. Find a formula for the mass of the sample that remains after t years. [5]

$$m(0) = 800 \text{ and } m(5) = \frac{1}{2}(800) = 400 \qquad 1$$

$$m(t) = m(0)e^{kt} = 800e^{kt} \qquad 1$$

$$\left.\begin{aligned} m(5) = 800e^{5k} &= 400 \\ e^{5k} &= \frac{1}{2} \\ k &= \frac{1}{5}\ln\left(\frac{1}{2}\right) \\ &= -\frac{1}{5}\ln(2) \end{aligned}\right\} \qquad 2$$

$$\boxed{m(t) = 800e^{-\frac{\ln(2)}{5}t}} \qquad 1$$

(b) Find the mass remaining after 10 years. [3]

$$\underset{1}{m(10)} = 800e^{-\overset{1}{\frac{\ln(2)}{5}}(10)} \overset{1}{=} 200.$$

6. Let $f(x) = x^3 - 3x^2$. Use calculus to find the following:

   (a) the critical numbers of $f(x)$ [2]

$$\begin{aligned} f'(x) = 3x^2 - 6x &= 0 \quad 1 \\ 3x(x-2) &= 0 \\ x = 0, x &= 2 \quad 1 \end{aligned}$$

   (b) the open intervals where $f(x)$ is increasing and where $f(x)$ is decreasing [4]

Interval	Test #	$f'(x) = 3x(x-2)$	$f(x)$ is ...	
$x < 0$	-1	(+)	increasing	1
$0 < x < 2$	1	(-)	decreasing	1
$x > 2$	3	(+)	increasing	1

Intervals of increase: $(-\infty, 0)$ and $(2, \infty)$

Intervals of decrease: $(0, 2)$ 1

   (c) the local maximum and minimum values of $f(x)$ [2]

Local max at $x = 0$ with the value $f(0) = 0$. 1

Local min at $x = 2$ with the value $f(2) = -4$. 1

(d) the open intervals where the graph of $f(x)$ is concave up and concave down [4]

1

$$f''(x) = 6x - 6 = 0 \implies x = 1$$

Interval	Test #	$f''(x) = 6x - 6$	$f(x)$ is concave ...	
$x < 1$	0	(-)	down	1
$x > 1$	2	(+)	up	1

Concave down: $(-\infty, 1)$

Concave up: $(1, \infty)$ 1

(e) any inflections points of $f(x)$ (1, -2) [1]

The graph changes from concave down to concave up at $x = 1$. Hence, $x = 1$ is an inflection point.

7. Use calculus to find the absolute maximum value of $f(x) = \sin(x) + 4$ on $[0, \pi]$. Make sure that you justify your final answer, which must be given exactly. [4]

Closed interval method:

$$f'(x) = \cos(x) = 0 \implies x = \frac{\pi}{2}.$$

Note $x = \pi/2$ above is the only point where $\cos(x) = 0$ in the interval $[0, \pi]$.

$$f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) + 4 = 1 + 4 = 5$$

Endpoints:

$$f(0) = \sin(0) + 4 = 0 + 4 = 4$$
$$f(\pi) = \sin(\pi) + 4 = 0 + 4 = 4$$

Hence the absolute maximum value of $f(x)$ is 5.

8. Use the **second derivative test** to determine if each critical point is a maximum, a minimum, or neither. $f(x) = xe^x$. [4]

$$f'(x) = (1)e^x + xe^x = e^x(1 + x) = 0 \implies x = -1 \text{ [since } e^x \neq 0]$$
$$f''(x) = e^x(1 + x) + e^x(1) = e^x(1 + x) + e^x$$
$$f''(-1) = e^{-1}(1 + (-1)) + e^{-1} > 0$$

Since $f''(-1) > 0$ [concave up] we have that $x = -1$ is a local min.

9. The kinetic energy of a particle is given by the equation $K = \frac{1}{2}mv^2$, where m is the mass of the particle and v is it's velocity. At what rate is the kinetic energy changing for a particle of mass 5 kg when its velocity is 10 m/sec and its acceleration is 3 m/sec^2? (Note: mass is constant - it does not change over time.) [5]

Given:

$$v = 10, \ a = \frac{dv}{dt} = 3 \text{ and } m = 5. \qquad 1$$

Since $m = 5$ is constant:

$$K = \frac{1}{2}(5)v^2 = \frac{5}{2}v^2. \qquad 1$$

Taking the derivative with respect to t yields

$$\frac{dK}{dt} = 5v\frac{dv}{dt}. \qquad 2$$

Hence,

$$\frac{dK}{dt} = 5(10)(3) = 150 \text{ N}\cdot\text{m/s} \qquad 1$$

10. A water tank has the shape of an inverted circular cone with base radius 3 m and height 6 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep. [5]

Using similar triangles we can setup a ratio:

$$\frac{r}{h} = \frac{3}{6} \implies r = \frac{1}{2}h \qquad 1$$

Substitute this into the volume formula:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h = \frac{1}{3}\pi\frac{1}{4}h^3 = \frac{\pi}{12}h^3. \qquad 1$$

Taking the derivative of the equation for V we just found:

$$\frac{dV}{dt} = \frac{3\pi}{12}h^2\frac{dh}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt} \qquad 2$$

We are given $\frac{dV}{dt} = 2$ and want to find $\frac{dh}{dt}$ when $h = 3$:

$$\begin{aligned}
\frac{dV}{dt} &= \frac{\pi}{4}h^2\frac{dh}{dt} \\
2 &= \frac{\pi}{4}(3)^2\frac{dh}{dt} \\
\frac{dh}{dt} &= \frac{8}{9\pi} \approx 0.2829 \text{ m/min} \qquad 1
\end{aligned}$$

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