Exam 1

Section W10 - Summer Session I 2016

June 10, 2016

Name: SOLUTIONS

PLEASE READ THE FOLLOWING INFORMATION.

- Please read each question carefully. Show **all** work clearly in the space provided. In order to receive full credit on a problem, solution methods must be complete, logical and understandable.
- Answers must be clearly labeled in the spaces provided after each question. Please cross out or fully erase any work that you do not want graded. The point value of each question is indicated after its statement.
- Give any numerical answers in exact form, not as approximations. For example, one-third is $\frac{1}{3}$, not .33 or .33333. And one-half of π is $\frac{1}{2}\pi$, not 1.57 or 1.57079.
- No books or other references are permitted. Turn smart phones, cell phones, and other electronic devices off and store them away.
- Calculators are allowed but you must show all your work in order to receive credit on the problem.
- If you finish early then you can hand in your exam early.

Part I Definitions and Concepts

1. What are three conditions for continuity? [3]

   (a) $f(a)$ exists 1

   (b) $\lim_{x \to a} f(x)$ exists 1

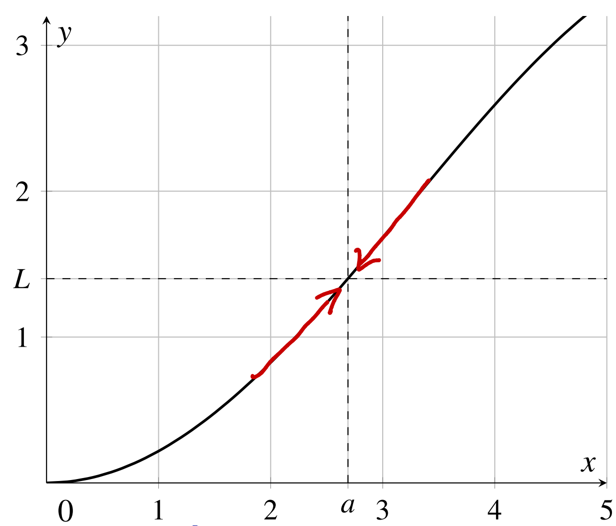
   (c) $\lim_{x \to a} f(x) = f(a)$ $\left[\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)\right]$ 1

2. What are two meanings of the derivative? [2]

   (a) • Slope of the tangent line 1
   • Limit of the slopes of secant lines

   (b) • Instantaneous rate of change of y with respect to x 1
   • If $f(x)$ represents the position of an object then $f'(x)$ is the object's velocity

3. Explain what the definition of the limit, $\lim_{x \to a} f(x) = L$, means. Add an appropriate illustration to the diagram to help support your explanation. [5]

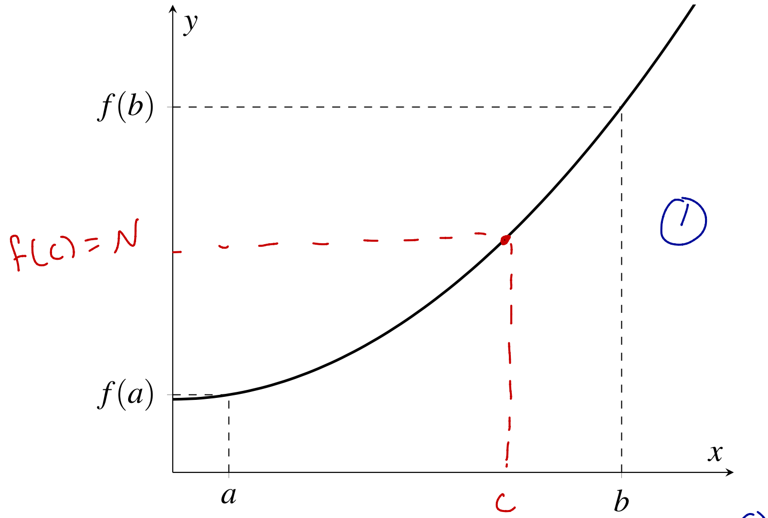


1

1 2 1

As x approaches a [from both the right and the left] the function $f(x)$ approaches L

4. Explain what the Intermediate Value Theorem (IVT) says. Use the diagram below to help support your explanation. (You do not need to give a precise statement. Just explain in English the gist of what the IVT says.) [5]



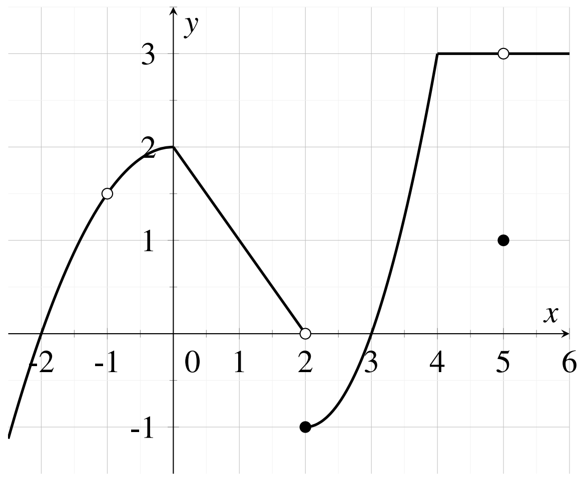
Loosely speaking, the intermediate value theorem says that on a closed interval $[a,b]$ a continuous function $f(x)$ attains all values (or "heights") between $f(a)$ and $f(b)$.

5. Fill in the following table. [3]

If $f(x)$ is...	then $f'(x)$ is...
Increasing	Positive
Decreasing	Negative
Correction: Max/Min	0

Part II Calculations

1. [9]



Use the graph of $f(x)$ to find the following limits, if they exist. If they do not exist, explain why.

(a) $\lim_{x\to -1} f(x)$ 1
1.5

(b) $\lim_{x\to 2} f(x)$
DNE because $\lim_{x\to 2^-} f(x) = 0 \neq \lim_{x\to 2^+} f(x) = -1$ 1

(c) $\lim_{x\to 5} f(x)$ 1
3

Use the graph to determine if $f(x)$ is continuous at the following points. If $f(x)$ is not continuous, explain why.

(a) $x = -1$
No, because $f(-1)$ is not defined.
1 1

(b) $x = 2$
No, because $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$ [see part (b)]
1 1

(c) $x = 5$
No, because $\lim_{x\to 5} f(x) = 3 \neq f(5) = 1$
1 1

2. Evaluate the following limits, **using algebraic methods** to simplify the expression before finding the limit. If the limit does not exist but goes to $+\infty$ or $-\infty$, indicate so. If the limit does not exist for any other reason, write **DNE** with a justification.

(a) $\displaystyle\lim_{x\to 1}\frac{x^2-6x+5}{x-1}$ [3]

$$\begin{aligned}\lim_{x\to 1}\frac{x^2-6x+5}{x-1} &= \lim_{x\to 1}\frac{(x-5)(x-1)}{(x-1)} \quad 1\\ &= \lim_{x\to 1}(x-5) \quad 1\\ &= -4 \quad 1\end{aligned}$$

(b) $\displaystyle\lim_{x\to 9}\frac{x-9}{\sqrt{x}-3}$ [4]

$$\begin{aligned}\lim_{x\to 9}\frac{x-9}{\sqrt{x}-3} &= \lim_{x\to 9}\frac{x-9}{\sqrt{x}-3}\left(\frac{\sqrt{x}+3}{\sqrt{x}+3}\right) \quad 1\\ &= \lim_{x\to 9}\frac{(x-9)(\sqrt{x}+3)}{(x-9)} \quad 1\\ &= \lim_{x\to 9}(\sqrt{x}+3) \quad 1\\ &= 6 \quad 1\end{aligned}$$

(c) $\displaystyle\lim_{x\to 0}\frac{(2+x)^2-4}{x}$ [4]

$$\begin{aligned}\lim_{x\to 0}\frac{(2+x)^2-4}{x} &= \lim_{x\to 0}\frac{4+4x+x^2-4}{x} \quad (1)\\ &= \lim_{x\to 0}\frac{4x+x^2}{x} \quad (1)\\ &= \lim_{x\to 0} 4+x \quad (1)\\ &= 4 \quad (1)\end{aligned}$$

(d) $\displaystyle\lim_{x\to 4}\frac{x+2}{5(4-x)}$ [4]

Notice that upon taking the limit you get $\displaystyle\lim_{x\to 4}\frac{x+2}{5(4-x)}=\frac{6}{0}$. (1) There is no algebra that can be done to simplify the expression any further. Therefore, the only option is to look at the left and right hand limits:

$$\lim_{x\to 4^-}\frac{x+2}{5(4-x)}=\frac{6}{\text{small positive \#}}=+\infty \quad (1)$$

$$\lim_{x\to 4^+}\frac{x+2}{5(4-x)}=\frac{6}{\text{small negative \#}}=-\infty \quad (1)$$

Hence, $\displaystyle\lim_{x\to 4^-}f(x)\neq\lim_{x\to 4^+}f(x)$ and so the limit does not exist. (1)

3. Let $f(x) = \begin{cases} 2-ax, & \text{if } x < 1 \\ a-4x & \text{if } x \geq 1 \end{cases}$ for a constant a.

(a) Compute $\lim_{x\to 1^-} f(x)$ in terms of a. [2]

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2-ax = 2-a$$

1 1

(b) Compute $\lim_{x\to 1^+} f(x)$ in terms of a. [2]

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} a-4x = a-4$$

1 1

(c) Find the value of a that makes $f(x)$ continuous at $x = 1$. [2]
In order for $f(x)$ to be continuous at $x = 1$ we need to have that

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x).$$

Hence, we must have

1 1

$$2-a = a-4 \implies 2a = 6 \implies a = 3$$

4. Let $f(x) = \dfrac{\sqrt{x+6x^2}}{2x-1}$.

(a) Find all the vertical asymptotes of the graph $y = f(x)$. [7]

The denominator is zero when $x = \dfrac{1}{2}$. This is the point where a vertical asymptote may be. Let's check:

$$\begin{aligned}
\lim_{x\to\frac{1}{2}^-} f(x) &= \lim_{x\to\frac{1}{2}^-} \frac{\sqrt{x+6x^2}}{2x-1} \quad 1 \\
1 \qquad &= \frac{\text{finite \#}}{\text{small negative \#}} \\
&= -\infty \quad 1
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to\frac{1}{2}^+} f(x) &= \lim_{x\to\frac{1}{2}^+} \frac{\sqrt{x+6x^2}}{2x-1} \quad 1 \\
1 \qquad &= \frac{\text{finite \#}}{\text{small positive \#}} \\
&= +\infty \quad 1
\end{aligned}$$

Therefore, there is indeed a vertical asymptote at $x = \dfrac{1}{2}$. 1

(b) Compute $\lim\limits_{x\to\infty} f(x)$, using **algebraic methods** to rewrite $f(x)$ so that the limit laws as $x \to \infty$ can be applied. [5]

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{\sqrt{x+6x^2}}{2x-1} = \frac{\infty}{\infty} \\
&= \lim_{x\to\infty} \frac{\sqrt{x+6x^2}}{2x-1}\left(\frac{\frac{1}{x}}{\frac{1}{x}}\right) \quad 1 \\
&= \lim_{x\to\infty} \frac{\sqrt{x+6x^2}}{2x-1}\left(\frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{x}}\right) \quad 1 \\
&= \lim_{x\to\infty} \frac{\sqrt{\frac{x+6x^2}{x^2}}}{2-1/x} \quad 1 \\
&= \lim_{x\to\infty} \frac{\sqrt{1/x+6}}{2-1/x} \quad 1 \\
&= \frac{\sqrt{0+6}}{2-0} \\
&= \frac{\sqrt{6}}{2} \quad 1
\end{aligned}$$

(c) Compute $\lim\limits_{x \to -\infty} f(x)$, using **algebraic methods** to rewrite $f(x)$ so that the limit laws as $x \to -\infty$ can be applied. [5]

$$\begin{aligned}
\lim_{x \to -\infty} f(x) &= \lim_{x \to -\infty} \frac{\sqrt{x+6x^2}}{2x-1} \\
&= \lim_{x \to -\infty} \frac{\sqrt{x+6x^2}}{2x-1}\left(\frac{\frac{1}{x}}{\frac{1}{x}}\right) \qquad 1 \\
&= \lim_{x \to -\infty} \frac{\sqrt{x+6x^2}}{2x-1}\left(\frac{-\frac{1}{\sqrt{x^2}}}{\frac{1}{x}}\right) \qquad 1 \\
&= \lim_{x \to -\infty} -\frac{\sqrt{\frac{x+6x^2}{x^2}}}{2-1/x} \qquad 1 \\
&= \lim_{x \to -\infty} -\frac{\sqrt{1/x+6}}{2-1/x} \qquad 1 \\
&= -\frac{\sqrt{0+6}}{2-0} \\
&= -\frac{\sqrt{6}}{2} \qquad 1
\end{aligned}$$

(d) What are all the horizontal asymptotes of the graph of $y = f(x)$. [2]

There are two vertical asymptotes: $y = \frac{\sqrt{6}}{2}$ and $y = -\frac{\sqrt{6}}{2}$.

1 1

5. Find the derivative of the function $f(x) = 2x^2 + 3x - 5$ at the point $x = a$ using the **limit definition of the derivative**. Your final answer should be expressed in terms of a. (No credit for using any other method other than the limit definition of the derivative.) [6]

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \qquad 1 \\
&= \lim_{h\to 0} \frac{\left[2(a+h)^2 + 3(a+h) - 5\right] - \left[2a^2 + 3a - 5\right]}{h} \qquad 1 \\
&= \lim_{h\to 0} \frac{2a^2 + 4ah + 2h^2 + 3a + 3h - 5 - 2a^2 - 3a + 5}{h} \qquad 1 \\
&= \lim_{h\to 0} \frac{4ah + 2h^2 + 3h}{h} \qquad 1 \\
&= \lim_{h\to 0} 4a + 2h + 3 \qquad 1 \\
&= 4a + 3 \qquad 1
\end{aligned}
$$

6. Given $f(x) = \dfrac{1}{x}$ answer the following:

   (a) Find the equation of the secant line to $f(x)$ passing through $(1, f(1))$ and $(2, f(2))$, writing the final answer in the form $y = mx + b$. [3]
   We are looking for the equation $y = mx + b$ where m is the slope of the secant line passing through the points $(1,1)$ and $(2,1/2)$.

$$m = \frac{f(2) - f(1)}{2-1} = \frac{\frac{1}{2} - \frac{1}{1}}{1} = -\frac{1}{2} \qquad 2$$

   Hence,

$$y - 1 = \left(-\frac{1}{2}\right)(x-1) \implies y = -\frac{1}{2}x + \frac{3}{2} \qquad 1$$

   (b) Find the equation of the tangent line to $f(x)$ passing through $(1, f(1))$, writing the final answer in the form $y = mx + b$. [4]
   We are looking for the equation $y = mx + b$ where m is the slope of the tangent line passing through the point $(1,1)$.

$$f'(x) = -\frac{1}{x^2} \qquad 2$$

$$m = f'(1) = -1. \qquad 1$$

   Hence,

$$y - 1 = (-1)(x-1) \implies y = -x + 2 \qquad 1$$

7. Find the derivatives of the following functions using the rules of differentiation.

(a) $f(x) = 7x^2 - 4\sqrt[3]{x}$ [3]

1 1 1

$$f(x) = 7x^2 - 4x^{1/3} \implies f'(x) = 14x - \frac{1}{3}4x^{1/3-1} = 14x - \frac{4}{3}x^{-2/3}$$

(b) $f(x) = x^3 e^x$ [3]

By the product rule,

$$\begin{aligned} f'(x) &= (3x^2)(e^x) + (x^3)(e^x) \\ &= x^2 e^x (3 + x) \end{aligned}$$

(c) $f(x) = \dfrac{x}{x^2-2}$ (simplify the numerator in final answer by combining like terms) [3]
By the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(1)(x^2-2)-(x)(2x)}{(x^2-2)^2} \quad 2 \\ &= \frac{x^2-2-2x^2}{(x^2-2)^2} \\ &= \frac{-x^2-2}{(x^2-2)^2} \quad 1 \end{aligned}$$

(d) $f(x) = \dfrac{e^x}{1-e^x}$ (simplify the numerator in final answer by combining like terms) [3]
By the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(e^x)(1-e^x)-(e^x)(-e^x)}{(1-e^x)^2} \quad 2 \\ &= \frac{e^x-e^{2x}+e^{2x}}{(1-e^x)^2} \\ &= \frac{e^x}{(1-e^x)^2} \quad 1 \end{aligned}$$

8. A ball thrown upwards has position $s(t) = 12t - 16t^2$ feet, where t is measured in seconds.

   (a) Determine the instantaneous velocity of the ball at time t, in ft/sec. [3]

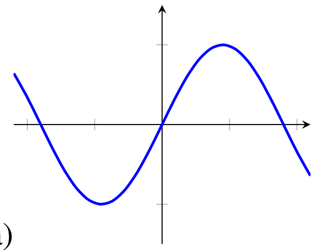
   $$v(t) = s'(t) = 12 - 32t \text{ ft/sec}$$

   (b) At what time t, in seconds, does the ball have an instantaneous velocity of 0 ft/s? [2]

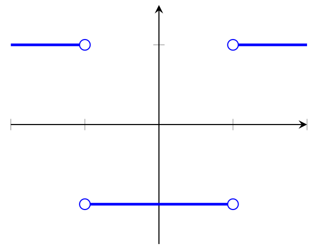
   $$v(t) = 12 - 32t = 0 \implies 32t = 12 \implies t = \frac{3}{8}$$

9. Match the graph of each function (a) - (c) with the graph of it's derivatives. [3]

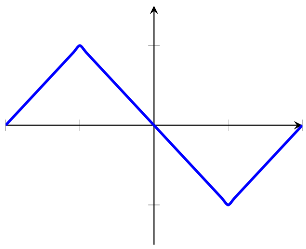
Functions Derivatives



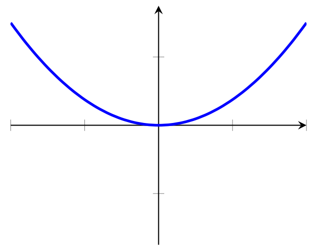
(a)



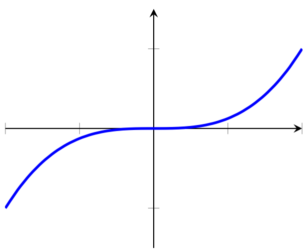
1 **b**



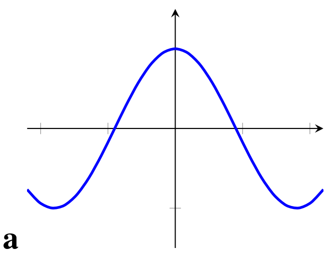
(b)



1 **c**



(c)



1 **a**