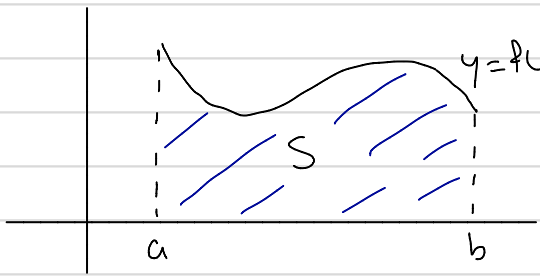
Chapter 5: Integrals

Section 5.1: Areas & Distances

The Area Problem

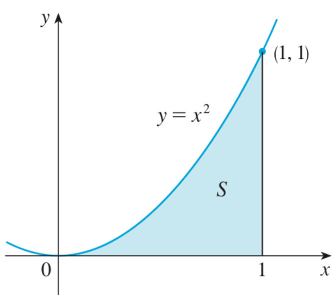


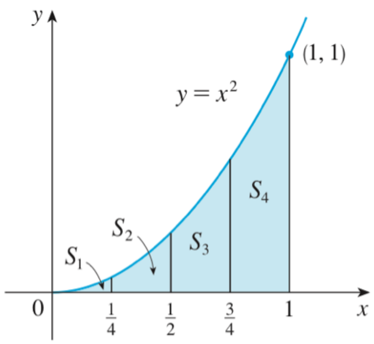
We want to find the area of the region S under the curve $y=f(x)$ from $x=a$ to $x=b$.

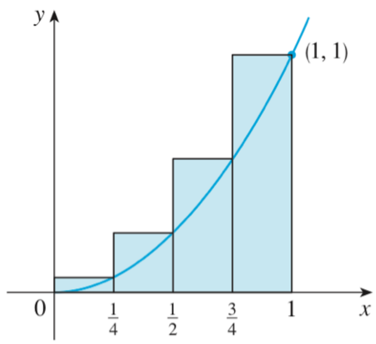
Idea: Try + use rectangles to approximate the area of S.

Ex. **EXAMPLE 1** Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1 (the parabolic region S illustrated in Figure 3).

Like we said, the idea is to split the region up into rectangles:







Divide up S into 4 strips

Approximate each strip w/ a rectangle whose height is the same as the right edge of the strip

Each rectangle has width 1/4 + height $f(x) = x^2$

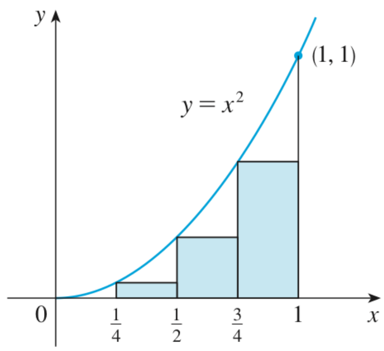
Now we just have to calculate the area of each rectangle, + add them all up: (Area of a rectangle = $w \cdot h = \frac{1}{4} \cdot f(x) = \frac{1}{4}x^2$)

$$R_4 = \left(\tfrac{1}{4}\right)\cdot\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{4}\right)\cdot\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)^2 + \left(\tfrac{1}{4}\right)(1)^2 = \frac{15}{32} \approx 0.46875$$

From the figure we see that:

$$\frac{1}{3} = \text{Actual Area} < R_4 = 0.46875$$

Alternatively, we can do the same thing using rectangles whose heights are the values of f at the left endpoints:



$$L_4 = \frac{1}{4}\cdot 0^2 + \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 = \frac{7}{32} \approx 0.21875$$

From the figure we see that:

$$\frac{1}{3} = \text{Actual Area} > L_4 \approx 0.21875$$

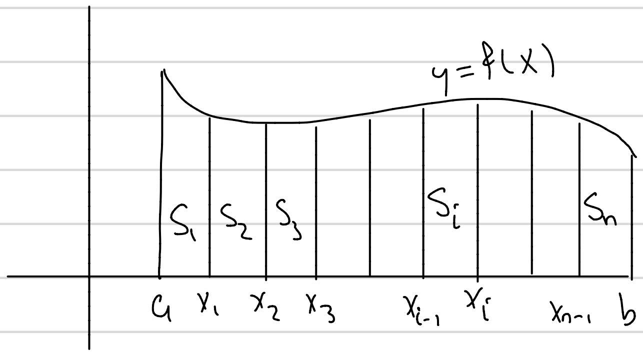
Notice that if I divide the region into more + more rectangles, we get better + better approximations.

* Show GIF ?

In other words,

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \frac{1}{3}.$$

Now let's apply the idea of the previous example to a general region S:



1) Divide S into n strips, $S_1, S_2, \ldots, S_n$, of equal width.

width of each interval: $\Delta x = \dfrac{b-a}{n}$

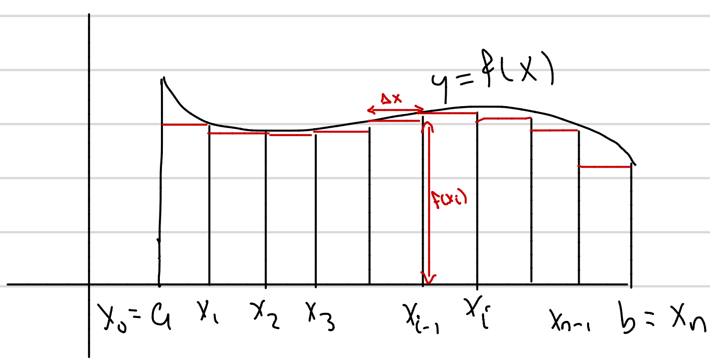
These strips divide $[a,b]$ into n subintervals

$$[a, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, b]$$

The right endpoints of the subintervals are

$$x_1 = a + \Delta x$$
$$x_2 = a + 2\Delta x$$
$$x_3 = a + 3\Delta x$$
$$\vdots$$

2) Approximate each strip with a rectangle of width Δx and height $f(x_i)$ (value of f at right endpoint)



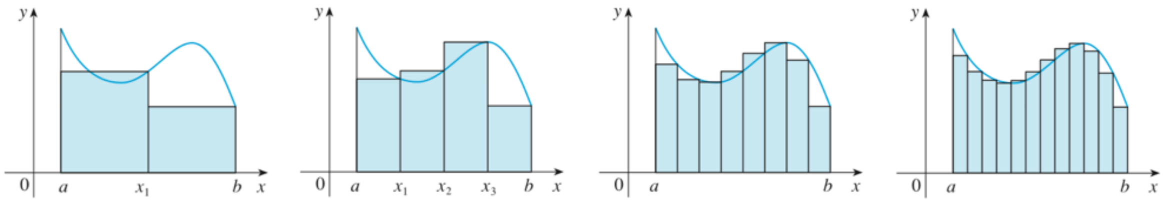
3) Area of each rectangle is

$$w \cdot h = \Delta x \cdot f(x_i)$$

4) Add up the area of all the rectangles:

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

* Notice again that the approximation gets better as $n \to \infty$.



(a) $n = 2$ (b) $n = 4$ (c) $n = 8$ (d) $n = 12$

Therefore, this is how we define the area of a region.

2 Definition The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

It can be shown that we get the same value using left endpoints:

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x]$$

Notation: $\sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$

This sort of sum is called a Riemann Sum

We can use this notation to rewrite the previous formulas:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x.$$

EXAMPLE 3 Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$.

(a) Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.

(b) Estimate the area by taking the sample points to be midpoints and using four subintervals and then ten subintervals.

SOLUTION

(a) Since $a = 0$ and $b = 2$, the width of a subinterval is

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

So $x_1 = 2/n$, $x_2 = 4/n$, $x_3 = 6/n$, $x_i = 2i/n$, and $x_n = 2n/n$. The sum of the areas of the approximating rectangles is

$$\begin{aligned} R_n &= f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x \\ &= e^{-x_1}\,\Delta x + e^{-x_2}\,\Delta x + \cdots + e^{-x_n}\,\Delta x \\ &= e^{-2/n}\left(\frac{2}{n}\right) + e^{-4/n}\left(\frac{2}{n}\right) + \cdots + e^{-2n/n}\left(\frac{2}{n}\right) \end{aligned}$$

According to Definition 2, the area is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2}{n}\left(e^{-2/n} + e^{-4/n} + e^{-6/n} + \cdots + e^{-2n/n}\right)$$

Using sigma notation we could write

$$A = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} e^{-2i/n}$$

Ex. cont.

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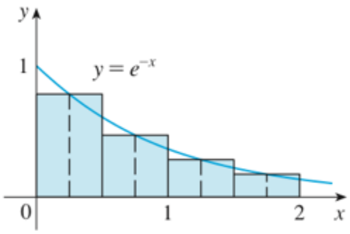


FIGURE 15

(b) With $n = 4$ the subintervals of equal width $\Delta x = 0.5$ are $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, and $[1.5, 2]$. The midpoints of these subintervals are $x_1^* = 0.25$, $x_2^* = 0.75$, $x_3^* = 1.25$, and $x_4^* = 1.75$, and the sum of the areas of the four approximating rectangles (see Figure 15) is

$$M_4 = \sum_{i=1}^{4} f(x_i^*)\,\Delta x$$

$$= f(0.25)\,\Delta x + f(0.75)\,\Delta x + f(1.25)\,\Delta x + f(1.75)\,\Delta x$$

$$= e^{-0.25}(0.5) + e^{-0.75}(0.5) + e^{-1.25}(0.5) + e^{-1.75}(0.5)$$

$$= \tfrac{1}{2}\left(e^{-0.25} + e^{-0.75} + e^{-1.25} + e^{-1.75}\right) \approx 0.8557$$

So an estimate for the area is

$$A \approx 0.8557$$

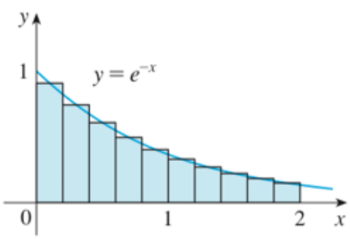


FIGURE 16

With $n = 10$ the subintervals are $[0, 0.2]$, $[0.2, 0.4]$, . . . , $[1.8, 2]$ and the midpoints are $x_1^* = 0.1$, $x_2^* = 0.3$, $x_3^* = 0.5$, . . . , $x_{10}^* = 1.9$. Thus

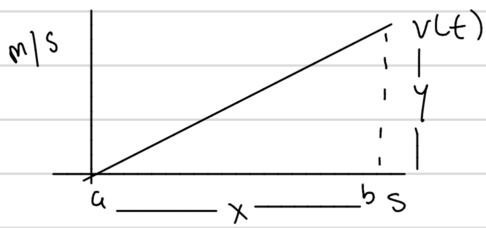
$$A \approx M_{10} = f(0.1)\,\Delta x + f(0.3)\,\Delta x + f(0.5)\,\Delta x + \cdots + f(1.9)\,\Delta x$$

$$= 0.2\left(e^{-0.1} + e^{-0.3} + e^{-0.5} + \cdots + e^{-1.9}\right) \approx 0.8632$$

From Figure 16 it appears that this estimate is better than the estimate with $n = 4$. ■

The Distance Problem

Fact: Distance travelled = Area under velocity graph



$A = \frac{1}{2} bh$

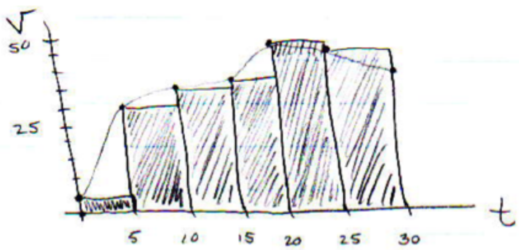
$= \frac{1}{2}(x\,s)(y\,m/s)$

$= \frac{1}{2}xy\ m$

Ex Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second interval. We take speedometer readings every five seconds and record them:

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	45	41

Method 1
using left-handed rectangles



distance = area ≈ sum of rectangle areas using left velocity as height of rectangle

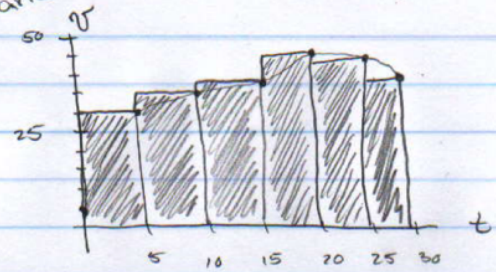
$\approx (5)(25) + (5)(31) + (5)(35) + (5)(43) + (5)(47) + (5)(46)$

≈ 1135 ft

an underestimate of the actual distance, or actual area

Method 2

using right-handed rectangles



distance $\approx (5)(37)+(5)(35)+(5)(43)+(5)(47)+(5)(46)+(5)(41)$

$\approx$ 1215 ft

an overestimate of actual area/ distance